PHYS 411

Discussion Class 1

7 September 2007

1. Is the cross product *associative*, i.e. is

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$$

   true for arbitrary vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$?

   If so, prove it; if not, provide a counterexample.

2. Find the transformation matrix R that describes a rotation by 120^0 about an axis from the origin through the point (1,1,1). The rotation is *clockwise* as you look down the axis toward the origin.

3. Compute the divergence of the vector function

$$\mathbf{v} = \frac{\hat{\mathbf{r}}}{r^2}$$

   using just the definition of divergence. Then check the validity of the *divergence theorem* by considering a spherical surface. The answer may surprise you...can you explain it?

4. Prove the following relations for Delta functions.

$$\begin{aligned} \delta(kx) &= \frac{1}{|k|}\delta(x) \qquad (k \text{ is a non} - \text{zero constant}) \\ x\frac{d}{dx}[\delta(x)] &= -\delta(x) \\ \frac{d\Theta}{dx} &= \delta(x) \qquad (\text{where } \Theta \text{ is the Heavyside step function}) \end{aligned}$$

Solutions

1. The triple cross product is *not* associative, in general. For example, suppose $\mathbf{A} = \mathbf{B}$ and $\mathbf{C} \perp \mathbf{A}$. Then $(\mathbf{B} \times \mathbf{C})$ points outward and $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ points down, as shown in Figure 1.1. Clearly, the magnitude $ABC \neq 0$. However, $(\mathbf{A} \times \mathbf{B}) = 0$ by construction, so $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \neq \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$.

2. A 120^0 rotation carries the z-axis into the $y(= \bar{z})$ axis, y into $x(= \bar{y})$, and x into $z(= \bar{x})$, as shown in Figure 1.2. So $\bar{A}_x = A_z,\ \bar{A}_y = A_x,\ \bar{A}_z = A_y$. Hence

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

3. By definition of divergence,

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{d}{dr}(r^2 v_r) = 0$$

So $\int_{\mathcal{V}} \nabla \cdot \mathbf{v} d\tau = 0$. However,

$$\oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} = \frac{1}{R^2} R^2 4\pi = 4\pi$$

This result is surprising, because, the vector field is obviously diverging away from the origin, still $\nabla \cdot \mathbf{v} = 0$ and the divergence theorem seems not to be satisfied! The answer is that $\nabla \cdot \mathbf{v} = 0$ everywhere *except* at the origin, where the field $\mathbf{v}$ blows up. In fact, $\nabla \cdot \mathbf{v}$ is *infinite* at origin, and zero elsewhere. So it can be written as

$$\nabla \cdot \mathbf{v} = 4\pi \delta^3(\mathbf{r})$$

which now satisfies the divergence theorem.

4. Two relations involving Delta functions are equal if

$$\int_{-\infty}^{\infty} f(x) D_1(x) dx = \int_{-\infty}^{\infty} f(x) D_2(x) dx$$

(a) Let $y \equiv kx \Rightarrow x = y/k$. Now

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\delta(kx)dx &= \begin{cases} \int_{-\infty}^{\infty} f\left(\frac{y}{k}\right)\delta(y)\frac{dy}{k} & (k > 0) \\ \int_{\infty}^{-\infty} f\left(\frac{y}{k}\right)\delta(y)\frac{dy}{k} & (k < 0) \end{cases} \\
&= \frac{1}{|k|}\int_{-\infty}^{\infty} f\left(\frac{y}{k}\right)\delta(y)dy \\
&= \frac{1}{|k|} f(0) \\
&= \frac{1}{|k|}\int_{-\infty}^{\infty} f(x)\delta(x)dx
\end{aligned}$$

Hence $\delta(kx) = \frac{1}{|k|}\delta(x)$.

(b)

$$\int_{-\infty}^{\infty} f(x)\left[x\frac{d}{dx}\delta(x)\right]dx = xf(x)\delta(x)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{d}{dx}[xf(x)]\delta(x)dx$$

The first term is zero, since $\delta(x) = 0$ at $\pm\infty$. So the integral is

$$-\int_{-\infty}^{\infty}\left[x\frac{df}{dx} + f\right]\delta(x)dx = -f(0) = -\int_{-\infty}^{\infty} f(x)\delta(x)dx$$

So $x\frac{d}{dx}\delta(x) = -\delta(x)$.

(c) The **Heavyside step function** is

$$\Theta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Hence

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\frac{d\Theta}{dx}dx &= f(x)\Theta(x)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{df}{dx}\Theta(x)dx \\
&= f(\infty) - \int_{0}^{\infty}\frac{df}{dx}dx = f(\infty) - [f(\infty) - f(0)] = f(0) \\
&= \int_{-\infty}^{\infty} f(x)\delta(x)dx
\end{aligned}$$

So $\frac{d\Theta}{dx} = \delta(x)$.