PHYS 411

Discussion Class 10

09 November 2007

1. A copper penny is placed on edge in a vertical magnetic field B=2 T. It is given a slight push to start it falling. Estimate how long it takes to fall. [The conductivity and density of Cu are 6×10^7 Ωm^{-1} and 9×10^3 kg m^{-3}, respectively. Take the radius of the penny, $r_0 = 0.01$ m, and the initial angle between the plane of the copper penny and the vertical axis, $\theta_0 = 0.1$ rad.]

   [**Hint:** Assume that in the falling process the *magnetic* torque is always in equilibrium with the *gravitational* torque.]

2. The Lorentz force law for a particle of mass m and charge q is

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

   (a) Show that if the particle moves in a *time-independent* electric field $\mathbf{E} = -\nabla\phi(x, y, z)$ and *any* magnetic field, then the energy $\frac{1}{2}m\mathbf{v}^2 + q\phi$ is a constant.

   (b) Suppose the particle moves along the x-axis in the electric field $\mathbf{E} = Ae^{-t/\tau}\hat{\mathbf{x}}$, where A and τ are both constants, with no magnetic field along x-axis. Given the initial conditions $x(0) = 0 = \dot{x}(0)$, find $x(t)$.

   (c) Using $x(t)$ found in (b), find whether $\frac{1}{2}m\mathbf{v}^2 - qxAe^{-t/\tau}$ is a constant. Indicate briefly your reasoning.

Solution

1. Let θ be the angle between the plane of the copper penny and the vertical axis. When we are considering a ring of radii r and $r+dr$, the magnetic flux crossing the area of the ring is

$$\phi(\theta) = \pi r^2 B \sin\theta$$

The induced emf in the ring is

$$\varepsilon = \left|\frac{d\phi}{dt}\right| = \pi r^2 B \cos\theta\dot{\theta}$$

and the induced current is

$$dI = \frac{\varepsilon}{R} = \frac{\pi r^2 B \cos\theta\dot{\theta}}{R}$$

where R is the resistance of the ring. Let h be the thickness of the penny. Then we have

$$R = \frac{2\pi r}{\sigma h dr}$$

where σ is the conductivity of Cu. Thus

$$dI = \frac{1}{2} B r \dot{\theta} \cos\theta \sigma h dr$$

The magnetic moment of the ring is

$$dm = \pi r^2 dI = \frac{1}{2}\pi r^3 B \dot{\theta}\cos\theta\sigma h dr$$

and the magnetic torque is

$$d\tau_m = |d\mathbf{m} \times \mathbf{B}| = dm B\cos\theta = \frac{1}{2}\pi r^3 B^2 \dot{\theta}\cos^2\theta\sigma h dr$$

If r_0 is the radius of the penny, then the magnetic torque on the whole penny is

$$\tau_m = \int d\tau_m = \frac{1}{2}\pi B^2\dot{\theta}\cos^2\theta\sigma h \int_0^{r_0} r^3 dr = \frac{1}{8}\pi B^2 \dot{\theta}\cos^2\theta\sigma h r_0^4 \tag{1}$$

The gravitational torque on the other hand is given by

$$\tau_g = mgr_0\sin\theta = \pi r_0^3 \rho g h \sin\theta \tag{2}$$

Assuming that in the falling process the magnetic torque is always in equilibrium with the gravitational torque, i.e. $\tau_m = \tau_g$, we get

$$dt = \frac{B^2 r_0 \sigma}{8 g\rho}\frac{\cos^2\theta}{\sin\theta}d\theta \tag{3}$$

As the penny starts falling from at $\theta = \theta_0$, the total falling time is

$$\begin{aligned} T &= \int dt = \frac{B^2 r_0 \sigma}{8 g \rho} \int_{\theta_0}^{\pi/2} \frac{\cos^2\theta}{\sin\theta} d\theta \\ &= \frac{B^2 \sigma r_0}{8 g \rho} \left|\cos\theta + \ln[\tan(\theta/2)]\right|_{\theta_0}^{\pi/2} \\ &= \frac{B^2 \sigma r_0}{8 g \rho} \left[-\cos\theta_0 + \frac{1}{2} \ln\left(\frac{1+\cos\theta_0}{1-\cos\theta_0} \right) \right] \end{aligned} \tag{4}$$

Using the given data, we have the estimate $T \approx 6.8$ s. We can conclude from this that the potential energy converts mainly into heat since the time required for falling in a strong magnetic field is much longer than the free-fall time ($T_{\text{free-fall}} \approx \sqrt{\frac{2r_0}{g}} \simeq 0.045$ s). This is because $T \propto \sigma$ and σ_{Cu} is very large (as Cu is a good conductor).

2. (a) We have to show that $\frac{1}{2}m\mathbf{v}^2 + q\phi$ is a constant, i.e. $\frac{d}{dt}\left[\frac{1}{2}m\mathbf{v}^2 + q\phi\right] = 0$.

$$\begin{aligned} \frac{d}{dt}\left[\frac{1}{2}m\mathbf{v}^2 + q\phi\right] &= m\mathbf{v}\cdot\dot{\mathbf{v}} + q\frac{d\phi}{dt} = m\mathbf{v}\cdot\dot{\mathbf{v}} + q\mathbf{v}\cdot\nabla\phi \\ &= \mathbf{v}\cdot(m\mathbf{v} + q\nabla\phi) = \mathbf{v}\cdot(m\dot{\mathbf{v}} - q\mathbf{E}) \end{aligned} \tag{5}$$

$$\begin{aligned} \text{Now} \quad & \mathbf{F} = m\dot{\mathbf{v}} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \\ \Rightarrow \quad & (m\dot{\mathbf{v}} - q\mathbf{E}) = q(\mathbf{v}\times\mathbf{B}) \\ \Rightarrow \quad & \mathbf{v}\cdot(m\dot{\mathbf{v}} - q\mathbf{E}) = q\mathbf{v}\cdot(\mathbf{v}\times\mathbf{B}) = 0 \end{aligned} \tag{6}$$

So $\frac{d}{dt}\left[\frac{1}{2}m\mathbf{v}^2 + q\phi\right] = 0$.

(b) The magnetic force $\mathbf{F}_m = q(\mathbf{v}\times\mathbf{B})$ is perpendicular to $\mathbf{v}$; hence if the particle moves in the x-direction, the magnetic force will not affect the x-component of the motion. With $\mathbf{E}$ in the x-direction the particle's motion will be confined in that direction. So

$$\begin{aligned} & m\ddot{x} = qE = qAe^{-t/\tau}, \\ \text{i.e.} \quad & m dv = qAe^{-t/\tau} dt. \\ \text{With} \quad & v(0) = 0, \quad mv = -qA\tau e^{-t/\tau} + qA\tau, \\ \text{or} \quad & dx = qA\tau(1 - e^{-t/\tau})\frac{dt}{m}. \\ \text{With} \quad & x(0) = 0, \quad x(t) = \frac{qA\tau}{m}\left[(t-\tau) + \tau e^{-t/\tau}\right] \end{aligned} \tag{7}$$

(c)

$$\begin{aligned} & \frac{1}{2}mv^2 - qxAe^{-t\tau} = \frac{1}{2}m\left[\frac{qA\tau}{m}(1-e^{-t/\tau})\right]^2 - \frac{q^2A^2\tau}{m}\left[(t-\tau) + \tau e^{-t/\tau}\right]e^{-t/\tau} \\ \Rightarrow \quad & \frac{d}{dt}\left(\frac{1}{2}mv^2 - qxAe^{-t/\tau}\right) \neq 0 \end{aligned}$$

This is because the electric field is not time-independent.