PHYS 411

Discussion Class 12

30 November 2007

1. Consider an electromagnetic wave of angular frequency ω in a medium containing a few free electrons of number density n_e.

   (a) Find the current density induced by the electric field (neglect interaction between electrons).

   (b) From Maxwell's equations write the differential equations for the spatial dependence of the wave in such a medium.

   (c) Find from this equation the necessary and sufficient condition that the electromagnetic wave propagates in this medium indefinitely (without attenuation).

   Hint: (a) You may use $\frac{\partial}{\partial t} \to -i\omega$ in the equation of motion of the free electron to find the current density. (b) Substitute this value of $\mathbf{j}$ in the corresponding Maxwell equation, and obtain the wave equation by combining it with other Maxwell equations. You can assume $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} = 0$ as the medium is charge free apart from a few free electrons. (c) You should find a spatial dependence of the form $\mathbf{E}(\mathbf{r}) \sim e^{i\mathbf{K}\cdot\mathbf{r}}$. The necessary and sufficient condition that the electromagnetic wave propagates in this medium indefinitely is that $\mathbf{K}$ is a real vector. Impose this condition to find the minimum frequency as a function of the number density of free electrons, n_e. This is called the *Plasma frequency* and is given by

$$\omega_P = \sqrt{\frac{n_e e^2}{\epsilon_0 m_e}}$$

2. X-rays which strike a metal surface at an angle of incidence to the normal greater than a critical angle θ_0 are totally reflected. Assuming that a metal contains n_e free electrons per unit volume, calculate θ_0 as a function of the angular frequency ω of the X-rays.

   Hint: The critical angle is given by Eq. 9.200, and the index of refraction of the metal is $n = \sqrt{1+\chi}$, where χ is the polarizability of the metal. Find χ assuming each electron to act as a Hertzian dipole. Here again you'll encounter the plasma frequency (See Problem 1).

Solution

1. (a) The equation of motion of an electron in the electromagnetic field of the wave is given by

$$m_e \frac{d\mathbf{v}}{dt} = -e\mathbf{E},$$

where the action of the magnetic field is neglected since it is of order vE/c, and $v \ll c$ for motion of free electrons. For a wave of angular frequency ω, $i\hbar\frac{\partial}{\partial t} \to \hbar\omega$, or $\frac{\partial}{\partial t} \to -i\omega$ and hence we get

$$\mathbf{v} = -\frac{ie}{m_e\omega}\mathbf{E}$$

Thus the current density is

$$\mathbf{j} = -n_e e\mathbf{v} = \frac{in_e e^2\mathbf{E}}{m_e\omega}$$

(b) The Maxwell equations are

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho, \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{B} &= \mu_0\mathbf{j} + \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}. \end{aligned}$$

So

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E}) &= \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} \\ &= -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\frac{\partial}{\partial t}\left(\mu_0\mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\right), \end{aligned} \quad (1)$$

as $c = (\mu_0\epsilon_0)^{-1/2}$. Assuming the medium to be charge free apart from a few free electrons, we can take $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} = 0$. Also

$$\frac{\partial \mathbf{j}}{\partial t} = -\frac{1}{i\omega}\frac{\partial^2 \mathbf{j}}{\partial t^2} = -\frac{n_e e^2}{m_e\omega^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Hence from Eq. 1 we get the wave equation

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\left(1 - \frac{\omega_P^2}{\omega^2}\right)\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0,$$

with $\omega_P^2 = \frac{n_e e^2}{m_e\epsilon_0}$. Putting

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0(\mathbf{r})e^{-i\omega t},$$

the spatial dependence of the wave equation can be written as

$$\nabla^2\mathbf{E}_0 + \frac{\omega^2}{c^2}\left(1 - \frac{\omega_P^2}{\omega^2}\right)\mathbf{E}_0 = 0 \quad (2)$$

(c) The solution of Eq. 2 is of the form

$$\mathbf{E}_0(\mathbf{r}) \sim e^{i\mathbf{K}\cdot\mathbf{r}}$$

Substitution in Eq. 2 yields

$$K^2c^2 = \omega^2 - \omega_P^2$$

The necessary and sufficient condition that the electromagnetic wave propagates in this medium indefinitely is that $\mathbf{K}$ is real, i.e.

$$\omega^2 > \omega_P^2, \quad \text{or} \quad \omega_{\min} = \omega_P \equiv \sqrt{\frac{n_e e^2}{m_e \epsilon_0}}$$

2. The equation of motion of a free electron in the field of the X-rays is given by

$$m_e \ddot{\mathbf{x}} = -e\mathbf{E}$$

We assume a solution of the form $\mathbf{x} = \mathbf{x}_0 e^{-i\omega t}$, for the electric field $\mathbf{E} = \mathbf{E}_0 e^{-i\omega t}$. This gives

$$m_e \omega^2 \mathbf{x} = e\mathbf{E}$$

Assuming each electron to act as a Hertzian dipole, the polarization vector of the metal is given by

$$\mathbf{P} = -n_e e\mathbf{x} = \chi\epsilon_0\mathbf{E},$$

giving the polarizability as

$$\chi = -\frac{n_e e^2}{m_e \epsilon_0 \omega^2} = -\frac{\omega_P^2}{\omega^2},$$

where ω_P is as defined in Problem 1. The index of refraction is given by

$$n = \sqrt{1+\chi} = \left(1 - \frac{\omega_P^2}{\omega^2}\right)^{1/2},$$

and hence the critical angle is (cf. Eq. 9.200)

$$\theta_0 = \sin^{-1}\left(1 - \frac{\omega_P^2}{\omega^2}\right)^{1/2}$$