1. Consider a star traveling with speed $v$ at an angle $\theta$ relative to our line of sight (see the figure). What is its apparent speed across the sky? What angle $\theta_0$ gives the maximum apparent speed? Show that the apparent speed can be much greater than the speed of light, even though $v$ itself is always less than $c$.

[Hint: Suppose the light signal from point $b$ reaches us at a time $\Delta t$ after the signal from $a$, and the star has meanwhile advanced a distance $\Delta s$ across the sky; then $u = \Delta s/\Delta t$ is the apparent speed. Show that as $v \to c$, $u \to \infty$.]

2. Let us consider a certain inertial frame $S$ in which the electric field $E$ and the magnetic field $B$ are neither parallel nor perpendicular, at a particular space-time point. Show that in a different inertial system $S'$, moving relative to $S$ with velocity $v$ given by

$$\frac{v}{1 + v^2/c^2} = \frac{E \times B}{B^2 + E^2/c^2},$$

the fields $\mathbf{E}$ and $\mathbf{B}$ are parallel at that point. Can there be any frame in which the fields are perpendicular?

[Hint: For simplicity, choose your axes so that $\mathbf{E} = (0, 0, E)$; $\mathbf{B} = (0, B \cos \phi, B \sin \phi)$. Use Eq. (12.108) to calculate $\mathbf{E}'$ and $\mathbf{B}'$ in the frame $S'$ moving at speed $v$ in the $x$-direction.]
Solution

1. 

\[ \begin{align*}
\text{light signal leaves } a \text{ at time } t_a' \; ; \; \text{arrives at earth at time } t_a = t_a' + d_a/c, \\
\text{light signal leaves } b \text{ at time } t_b' \; ; \; \text{arrives at earth at time } t_b = t_b' + d_b/c.
\end{align*} \]

Here \(d_a\) is the distance from \(a\) to earth, and \(d_b\) from \(b\) to earth. So

\[ \Delta t = t_b - t_a = t_b' - t_a' + \frac{d_b - d_a}{c} = \Delta t' + \left(\frac{-v\Delta t' \cos \theta}{c}\right) = \Delta t' \left(1 - \frac{v}{c} \cos \theta\right) \]

\[ \Delta s = v\Delta t' \sin \theta = \frac{v \sin \theta \Delta t}{1 - \frac{v}{c} \cos \theta} \]

\[ \Rightarrow u = \frac{\Delta s}{\Delta t} = \frac{v \sin \theta}{1 - \frac{v}{c} \cos \theta} \]

To find the angle at which \(u\) is maximum,

\[ 0 = \frac{du}{d\theta} = v \left[\left(1 - \frac{v}{c} \cos \theta\right) (\cos \theta) - \sin \theta \left(\frac{v}{c} \sin \theta\right)\right]\]

\[ \Rightarrow \left(1 - \frac{v}{c} \cos \theta\right) \cos \theta = \frac{v}{c} \sin^2 \theta \]

\[ \Rightarrow \cos \theta = \frac{v}{c}, \text{ or} \]

\[ \theta_0 = \cos^{-1}(v/c) \]

At this angle,

\[ u_0 = \frac{v \sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v^2}{c^2}} = \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} \]

As \(v \to c\), \(u \to \infty\), even though \(v < c\).

2. We choose the axes such that \(E\) points in the \(z\) direction and \(B\) in the \(yz\) plane:

\[ E = (0, 0, E); \quad B = (0, B \cos \phi, B \sin \phi). \]

Using Eq. (12.108), the transformed fields in the \(\mathbf{\bar{F}}\) frame moving at speed \(v\) in the \(x\) direction:

\[ \mathbf{\bar{E}} = (0, -\gamma v B \sin \phi, \gamma(E + v B \cos \phi)); \quad \mathbf{\bar{B}} = (0, \gamma(B \cos \phi + \frac{v}{c^2} E), \gamma B \sin \phi). \]

\(E \parallel B\) provided

\[ \frac{E_y}{B_y} = \frac{E_z}{B_z}, \]

or

\[ \frac{-\gamma v B \sin \phi}{\gamma(B \cos \phi + \frac{v}{c^2} E)} = \frac{\gamma(E + v B \cos \phi)}{\gamma B \sin \phi}, \]

or

\[ -\gamma v B^2 \sin^2 \phi = \left(B \cos \phi + \frac{v}{c^2} E\right)(E + v B \cos \phi), \]

or

\[ \frac{v}{1 + \frac{v^2}{c^2}} = -\frac{EB \cos \phi}{B^2 + E^2/c^2}. \]
Now $\mathbf{E} \times \mathbf{B} = -EB \cos \phi \hat{x}$. So

$$\frac{\mathbf{v}}{1 + v^2/c^2} = \frac{\mathbf{E} \times \mathbf{B}}{B^2 + E^2/c^2}$$

There can be no frame in which $\mathbf{E} \perp \mathbf{B}$, for $(\mathbf{E} \cdot \mathbf{B})$ is invariant, and since it is not zero in $\mathcal{S}$ it can’t be zero in $\mathcal{S}$. 

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