PHYS 411

Discussion Class 3

21 September 2007

1. Consider two concentric spherical shells, of radii a and b. Suppose the inner one carries a charge $+q$, and the outer one a charge $-q$ (both of them uniformly distributed over the surface). Calculate the electrostatic energy of this configuration.

2. A parallel-plate capacitor is charged to a potential V and then disconnected from the charging circuit. How much work is done by slowly changing the separation of the plates from d to $d' \neq d$? (Assume the plates to be circular with radius $r \gg d$.)

3. Given two plane-parallel electrodes, with separation d, and at voltages 0 and V_0, find the current density if an unlimited supply of electrons at rest is supplied to the lower potential electrode (neglecting collisions between the electrons).

Solutions

1. $W = \frac{\epsilon_0}{2}\int E^2 d\tau$. $\quad E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \quad (a < r < b)$, zero elsewhere. Hence

$$W = \frac{\epsilon_0}{2}\left(\frac{q}{4\pi\epsilon_0}\right)^2 \int_a^b \left(\frac{1}{r^2}\right)^2 4\pi r^2 dr = \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

2. Neglecting edge effects, the capacitance of the parallel-plate capacitor is $C = \frac{\epsilon_0 \pi r^2}{d}$ and the stored energy is $W = \frac{1}{2}CV^2$. As the charges on the plates, $Q = \pm CV$, do not vary with the separation, we have

$$V' = \frac{C}{C'}V$$

Thus the energy stored when the separation is d' is

$$W' = \frac{1}{2}C'\left(\frac{C}{C'}V\right)^2 = \frac{1}{2}\frac{C^2}{C'}V^2$$

Thus the change of the energy stored in the capacitor is

$$\Delta W = W' - W = \frac{1}{2}CV^2\left(\frac{C}{C'} - 1\right) = \frac{1}{2}CV^2\left(\frac{d'}{d} - 1\right) = \frac{\epsilon_0 \pi r^2 (d' - d)V^2}{2d^2}$$

Therefore, the work done in changing the separation from d to d' is $\frac{\epsilon_0 \pi r^2 (d'-d)V^2}{2d^2}$.

3. Let us choose the x-axis perpendicular to the plates as shown in Figure 3.1. Both the charge and current density are functions of x. In the steady state

$$\frac{d\mathbf{j}(x)}{dx} = 0, \quad \text{or} \quad \mathbf{j} = -j_0\hat{\mathbf{x}},$$

where j_0 is the constant to be determined. Let $v(x)$ be the velocity of the electrons. Then the charge density is (from the continuity equation)

$$\rho(x) = -\frac{j_0}{v(x)}$$

The potential satisfies the **Poisson equation**

$$\frac{d^2V(x)}{dx^2} = -\frac{\rho(x)}{\epsilon_0} = \frac{j_0}{\epsilon_0 v(x)} \tag{1}$$

Using the energy relation $\frac{1}{2}mv^2(x) = eV$, we get

$$\frac{d^2V(x)}{dx^2} = \frac{j_0}{\epsilon_0}\sqrt{\frac{m}{2eV(x)}} \tag{2}$$

To solve this differential equation, we take $u = \frac{dV}{dx}$. Then

$$\frac{d^2V}{dx^2} = \frac{du}{dx} = \frac{du}{dV}\frac{dV}{dx} = u\frac{du}{dV},$$

and the differential equation 2 becomes

$$udu = AV^{-1/2}dV,$$

where $A = \frac{j_0}{\epsilon_0}\sqrt{\frac{m}{2e}}$. At $x = 0,\ V = 0$ and also $u = \frac{dV}{dx} = 0$ as the electrons are at rest there. Hence we get the solution

$$\frac{1}{2}u^2 = 2AV^{1/2}, \quad \text{or} \quad V^{-1/4}dV = 2A^{1/2}dx$$

Using the boundary conditions $V = 0$ at $x = 0$ and $V = V_0$ at $x = d$, we get the solution

$$\frac{4}{3}V_0^{3/4} = 2A^{1/2}d = 2\left(\frac{j_0}{\epsilon_0}\sqrt{\frac{m}{2e}}\right)^{1/2} d$$

$$\Rightarrow j_0 = \frac{4\epsilon_0 V_0}{9d^2}\sqrt{\frac{2eV_0}{m}}$$

Finally the current density is given by

$$\mathbf{j} = -j_0\hat{\mathbf{x}} = -\frac{4\epsilon_0 V_0}{9d^2}\sqrt{\frac{2eV_0}{m}}\hat{\mathbf{x}}$$