## PHYS 411

## Discussion Class 4

28 September 2007

1. An electric field $\mathbf{E}(x, y, z)$ has the form

$$
E_{x}=a x, \quad E_{y}=0, \quad E_{z}=0
$$

where $a$ is a constant. What is the charge density? How do you account for the fact that the field points in a particular direction, when the charge density is uniform?
2. All of electrostatics basically follows from the inverse-square law, together with the principle of superposition. An analogous theory can therefore be constructed for Newton's law of gravitation.
(a) What is the gravitational energy of a sphere, of mass $M$ and radius $R$, assuming the density is uniform? (Hint: Use the expression for electrostatic energy of a uniformly charged sphere.)
(b) Use your result to estimate the gravitational energy of the sun. (Useful numbers: $G=6.67 \times 10^{-11} \mathrm{~N} \mathrm{~m}^{2} / \mathrm{kg}^{2}, M_{\text {sun }}=2 \times 10^{30} \mathrm{~kg}, R_{\text {sun }}=7 \times 10^{8} \mathrm{~m}$.)
(c) Note that this energy is negative - masses attract, whereas (like) charges repel. As the matter "falls in", to create the sun, the potential energy is converted into other forms, and is subsequently released in the form of radiation. The sun radiates at a rate of $3.86 \times 10^{26} \mathrm{~W}$; if all this came from gravitational energy, how long would the sun last?
3. Find the average potential over a spherical surface of radius $R$ due to a point charge $q$ located (a) outside, (b) inside of the sphere. Show that, in general,

$$
V_{\mathrm{ave}}=V_{\text {center }}+\frac{Q_{\mathrm{enc}}}{4 \pi \epsilon_{0} R},
$$

where $V_{\text {center }}$ is the potential at the center of the sphere due to all the external charges, and $Q_{\text {enc }}$ is the total enclosed charge. (Note: This result implies that $V$ can have no local maxima or minima; the extreme values of $V$ must occur at the boundaries.)

## Solutions

1. $\rho=\epsilon_{0} \nabla \cdot \mathbf{E}=\epsilon_{0} \frac{\partial}{\partial x}(a x)=\epsilon_{0} a$ (constant everywhere). The same charge density would be compatible (as far as Gauss's law is concerned) with $\mathbf{E} \frac{a}{3} \mathbf{r}$, for instance. The point is that Gauss's law (and $\nabla \times \mathbf{E}=0$ ) by themselves do not determine the field uniquely - like any differential equations, they must be supplemented by appropriate boundary conditions. Ordinarily, we impose them almost subconsciously (e.g. $E$ must go to zero far from the source charge) - or we appeal to symmetry to resolve the ambiguity (e.g. the field must be the same in magnitude on both sides of an infinite plane of surface charge). But in this case there are no natural boundary conditions, and no persuasive symmetry conditions, to fix the answer. The question is therefore ill-posed: it does not give us sufficient information to determine the charge density. (Note: Incidentally, it won't help to appeal to Coulomb's law either: $\mathbf{E}=\frac{1}{4 \pi \epsilon_{0}} \int \rho \frac{\hat{\mathrm{r}}}{r^{2}} d \tau$ - the integral is hopelessly indefinite, in this case.)
2. We compare Newton's law of universal gravitation to Coulomb's law:

$$
\mathbf{F}=-G \frac{m_{1} m_{2}}{r^{2}} \hat{\mathbf{r}} ; \quad \mathbf{F}=\frac{1}{4 \pi \epsilon_{0}} \frac{q_{1} q_{2}}{r^{2}} \hat{\mathbf{r}} .
$$

Evidently $\frac{1}{4 \pi \epsilon_{0}} \rightarrow G$ and $q \rightarrow m$ in case of gravitation.
(a) As we have already seen before, the electrostatic potential energy of a uniformly charged sphere is $\frac{1}{4 \pi \epsilon_{0}} \frac{3}{5} \frac{q^{2}}{R}$. Hence in analogy, the gravitational energy of a sphere of mass $M$ is therefore

$$
W_{\text {grav }}=-\frac{3}{5} \frac{G M^{2}}{R}
$$

(b) Substituting the numerical values of $G, M_{\text {sun }}$ and $R_{\text {sun }}$, we get

$$
W_{\text {sun }}=-2.28 \times 10^{41} \mathrm{~J}
$$

(c) At the current rate $P=3.86 \times 10^{26} \mathrm{~W}$, the energy would be dissipated in a time

$$
t=\frac{W}{P}=5.90 \times 10^{14} \mathrm{~s}=1.87 \times 10^{7} \text { years }
$$

(Note: The sun is in fact much older than this, about $5 \times 10^{9}$ years old. So evidently gravitational collapse is not the source of its power.)
3. We can center the sphere at the origin and choose coordinates so that $q$ lies on the $z$-axis (Figure 4.1). The potential at a point on the surface is given by

$$
V=\frac{1}{4 \pi \epsilon_{0}} \frac{q}{r}
$$

where $r^{2}=z^{2}+R^{2}-2 z R \cos \theta$. The average value of $V$ is given by

$$
\begin{align*}
V_{\text {ave }} & =\frac{1}{4 \pi R^{2}} \oint_{\text {sphere }} V d a \\
& =\frac{1}{4 \pi R^{2}} \frac{q}{4 \pi \epsilon_{0}} \int\left[z^{2}+R^{2}-2 z R \cos \theta\right]^{-1 / 2} R^{2} \sin \theta d \theta d \phi \\
& =\left.\frac{q}{4 \pi \epsilon_{0}} \frac{1}{2 z R} \sqrt{z^{2}+R^{2}-2 z R \cos \theta}\right|_{0} ^{\pi} \tag{1}
\end{align*}
$$

For an outside point $(z>R)$, Eq 1 gives

$$
\begin{equation*}
V_{\mathrm{ave}}=\frac{q}{4 \pi \epsilon_{0}} \frac{1}{2 z R}[(z+R)-(z-R)]=\frac{1}{4 \pi \epsilon_{0}} \frac{q}{z} \tag{2}
\end{equation*}
$$

which is precisely the potential due to $q$ at the center of the sphere. By superposition principle, the average potential due to any collection of external charges over the sphere is equal to the net potential they produce at the center.
Now for an inside point $(z<R)$, Eq 1 gives

$$
\begin{equation*}
V_{\mathrm{ave}}=\frac{q}{4 \pi \epsilon_{0}} \frac{1}{2 z R}[(R+z)-(R-z)]=\frac{1}{4 \pi \epsilon_{0}} \frac{q}{R} \tag{3}
\end{equation*}
$$

By superposition principle, if there is more than one charge inside the sphere, the average potential due to the interior charges is $\frac{1}{4 \pi \epsilon_{0}} \frac{Q_{\text {enc }}}{R}$.
Combining both the results (Eqs. 2 and 3), we get

$$
V_{\mathrm{ave}}=V_{\mathrm{center}}+\frac{Q_{\mathrm{enc}}}{4 \pi \epsilon_{0} R}
$$

