## PHYS 411

## Discussion Class 5

05 October 2007

1. Two infinite parallel grounded conducting planes are held a distance $a$ apart. A point charge $q$ is placed in the region between them, a distance $x$ from one plate. Find the force exerted on $q$. Check that your answer is correct for the special cases $a \rightarrow \infty$ and $x=a / 2$.
[Hint: Find out the correct image configuration. The positive image charge forces should cancel in pairs, leaving a net force of the negative image charges only.]
2. A charge $+q$ is distributed uniformly along the $z$-axis from $z=-a$ to $z=+a$. Find out the electric potential at a point $\mathbf{r}$ for $r>a$.
[Hint: Use multipole expansion

$$
V(\mathbf{r})=\frac{1}{4 \pi \epsilon_{0}} \sum_{n=0}^{\infty} \frac{1}{r^{(n+1)}} \int\left(r^{\prime}\right)^{n} P_{n}\left(\cos \theta^{\prime}\right) \rho\left(\mathbf{r}^{\prime}\right) d \tau^{\prime}
$$

and substitute $\rho d \tau \rightarrow \lambda d z, r^{\prime} \rightarrow z$. Then perform the $z$-integral only.]
3. The potential on the axis of a uniformly charged disk, of radius $R$ and with surface charge density $\sigma$, is given by

$$
V(r, 0)=\frac{\sigma}{2 \epsilon_{0}}\left[\sqrt{r^{2}+R^{2}}-r\right] \quad \text { (cf. Prob. 2.25c, Griffiths) }
$$

(a) Use this to evaluate the first three terms in the expansion

$$
V(r, \theta)=\sum_{l=0}^{\infty} \frac{B_{l}}{r_{l}^{l+1}} P_{l}(\cos \theta)
$$

for the potential of the disk at points off the axis, assuming $r>R$.
[Hint: Expand $V(r, 0)$ in powers of $r$ using the fact that $P_{l}(1)=1$.]
(b) Find the potential for $r<R$ by the same method, using the expansion

$$
V(r, \theta)=\sum_{l=0}^{\infty} A_{l} r^{l} P_{l}(\cos \theta)
$$

[Hint: You must break the interior region up into two hemispheres, above and below the disk as the coefficients $A_{l}$ may not be the same in both hemispheres.]

## Solution

1. The image configuration is shown in Figure 1. It is clear from the positions of image charges that the positive image charge forces cancel in pairs. The net force of the negative image charges (in the direction of $+x$ ) is:

$$
\begin{aligned}
F= & \frac{q^{2}}{4 \pi \epsilon_{0}}\left\{\frac{1}{[2(a-x)]^{2}}+\frac{1}{[2 a+2(a-x)]^{2}}+\frac{1}{[4 a+2(a-x)]^{2}}+\ldots\right. \\
& \left.-\frac{1}{(2 x)^{2}}-\frac{1}{(2 a+2 x)^{2}}-\frac{1}{(4 a+2 x)^{2}}-\ldots\right\} \\
& =\frac{q^{2}}{4 \pi \epsilon_{0}} \frac{1}{4}\left\{\left[\frac{1}{(a-x)^{2}}+\frac{1}{(2 a-x)^{2}}+\frac{1}{(3 a-x)^{2}}+\ldots\right]-\left[\frac{1}{x^{2}}+\frac{1}{(a+x)^{2}}+\frac{1}{(2 a+x)^{2}}+\ldots\right]\right\}
\end{aligned}
$$



Figure 1: Image configuration

When $a \rightarrow \infty$ (i.e. $a \gg x$ ), only the $\frac{1}{x^{2}}$ term survives:

$$
F=-\frac{1}{4 \pi \epsilon_{0}} \frac{q^{2}}{(2 x)^{2}}
$$

which is the same as for only one plate (cf. Eq. 3.12).
When $x=a / 2$, the two parts in parentheses exactly cancel, as expected by symmetry:
$F=\frac{q^{2}}{4 \pi \epsilon_{0}} \frac{1}{4}\left\{\left[\frac{1}{(a / 2)^{2}}+\frac{1}{(3 a / 2)^{2}}+\frac{1}{(5 a / 2)^{2}}+\ldots\right]-\left[\frac{1}{(a / 2)^{2}}+\frac{1}{(3 a / 2)^{2}}+\frac{1}{(5 a / 2)^{2}}+\ldots\right]\right\}=0$
2. We use the multipole expansion

$$
V(\mathbf{r})=\frac{1}{4 \pi \epsilon_{0}} \sum_{n=0}^{\infty} \frac{1}{r^{(n+1)}} \int\left(r^{\prime}\right)^{n} P_{n}\left(\cos \theta^{\prime}\right) \rho\left(\mathbf{r}^{\prime}\right) d \tau^{\prime}
$$

with $\rho d \tau \rightarrow \lambda d z=\frac{Q}{2 a} d z$, and $r^{\prime} \rightarrow z:$

$$
V(\mathbf{r})=\frac{1}{4 \pi \epsilon_{0}} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int_{-a}^{a} z^{n} P_{n}(\cos \theta) \frac{Q}{2 a} d z
$$

Now $\theta^{\prime}$, being the angle between $\mathbf{r}$ and $z$, is constant for a given $\mathbf{r}$, and hence, the integral becomes trivial:

$$
\frac{Q}{2 a} P_{n}(\cos \theta) \int_{-a}^{a} z^{n} d z=\left.\frac{Q}{2 a} P_{n}(\cos \theta) \frac{z^{n+1}}{n+1}\right|_{-a} ^{a}=\left\{\begin{array}{cc}
\frac{Q}{2 a} P_{n}(\cos \theta) \frac{2 a^{n+1}}{n+1} & \text { for } n \text { even } \\
0 & \text { for } n \text { odd }
\end{array}\right\}
$$

Therefore

$$
V(r)=\frac{Q}{4 \pi \epsilon_{0}} \frac{1}{r} \sum_{n=0,2,4, \ldots}\left[\frac{1}{n+1}\left(\frac{a}{r}\right)^{n} P_{n}(\cos \theta)\right]
$$

3. (a)

$$
\begin{aligned}
V(r, \theta) & =\sum_{l=0}^{\infty} \frac{B_{l}}{r^{l+1}} P_{l}(\cos \theta) \quad(r>R), \\
\text { so } V(r, 0) & =\sum_{l=0}^{\infty} \frac{B_{l}}{r^{l+1}} P_{l}(1)=\sum_{l=0}^{\infty} \frac{B_{l}}{r^{l+1}}=\frac{\sigma}{2 \epsilon_{0}}\left[\sqrt{r^{2}+R^{2}}-r\right] .
\end{aligned}
$$

Since $r>R$ in this region,

$$
\begin{aligned}
\sqrt{r^{2}+R^{2}} & =r\left[1+\left(\frac{R}{r}\right)^{2}\right]^{1 / 2}=r\left[1+\frac{1}{2}\left(\frac{R}{r}\right)^{2}-\frac{1}{8}\left(\frac{R}{r}\right)^{4}+\ldots-1\right] \\
& =\frac{\sigma}{2 \epsilon_{0}}\left(\frac{R^{2}}{2 r}-\frac{R^{4}}{8 r^{3}}+\ldots\right)
\end{aligned}
$$

Comparing like powers of $r$, we get $B_{0}=\frac{\sigma R^{2}}{4 \epsilon_{0}}, B_{1}=0, B_{2}=-\frac{\sigma R^{4}}{16 \epsilon_{0}}, \ldots$. Therefore for $r>R$,

$$
\begin{aligned}
V(r, \theta) & =\frac{\sigma R^{2}}{4 \epsilon_{0}}\left[\frac{1}{r}-\frac{R^{2}}{4 r^{3}} P_{2}(\cos \theta)+\ldots\right] \\
& =\frac{\sigma R^{2}}{4 \epsilon_{0} r}\left[1-\frac{1}{8}\left(\frac{R}{r}\right)^{2}\left(3 \cos ^{2} \theta-1\right)+\ldots\right]
\end{aligned}
$$

(b)

$$
V(r, \theta)=\sum_{l=0}^{\infty} A_{l} r^{l} P_{l}(\cos \theta) \quad(r<R)
$$

In the northern hemisphere $(0 \leq \theta \leq \pi / 2)$, we go for $\theta=0 \Rightarrow P_{l}(1)=1$. So

$$
V(r, 0)=\sum_{l=0}^{\infty} A_{l} r^{l}=\frac{\sigma}{2 \epsilon_{0}}\left[\sqrt{r^{2}+R^{2}}-r\right]
$$

Since $r<R$ in this region,

$$
\sqrt{r^{2}+R^{2}}=R\left[1+\left(\frac{r}{R}\right)^{2}\right]^{1 / 2}=R\left[1+\frac{1}{2}\left(\frac{r}{R}\right)^{2}-\frac{1}{8}\left(\frac{r}{R}\right)^{4}+\ldots\right]
$$

Therefore

$$
\sum_{l=0}^{\infty} A_{l} r^{l}=\frac{\sigma}{2 \epsilon_{0}}\left[R+\frac{r^{2}}{2 R}-\frac{r^{4}}{8 R^{3}}+\ldots-r\right]
$$

Comparing like powers: $A_{0}=\frac{\sigma}{2 \epsilon_{0}} R, A_{1}=-\frac{\sigma}{2 \epsilon_{0}}, A_{2}=\frac{\sigma}{2 \epsilon_{0} R}, \ldots$, so for $r<R$, northern hemisphere,

$$
\begin{aligned}
V(r, \theta) & =\frac{\sigma}{2 \epsilon_{0}}\left[R-r P_{1}(\cos \theta)+\frac{1}{2 R} P_{2}(\cos \theta)+\ldots\right] \\
& =\frac{\sigma R}{2 \epsilon_{0}}\left[1-\left(\frac{r}{R}\right) \cos \theta+\frac{1}{4}\left(\frac{r}{R}\right)^{2}\left(3 \cos ^{2} \theta-1\right)+\ldots\right]
\end{aligned}
$$

In the southern hemisphere we'll have to go for $\theta=\pi$, using $P_{l}(-1)=(-1)^{l}$.

$$
V(r, \pi)=\sum_{l=0}^{\infty}(-1)^{l} \tilde{A}_{l} r^{l}=\frac{\sigma}{2 \epsilon_{0}}\left[\sqrt{r^{2}+R^{2}}-r\right]
$$

The only difference between $\tilde{A}_{l}$ and $A_{l}$ is the sign of $\tilde{A}_{1}: \tilde{A}_{1}=+\frac{\sigma}{2 \epsilon_{0}}, \tilde{A}_{0}=$ $A_{0}, \tilde{A}_{2}=A_{2}$. So for $r<R$, southern hemisphere,

$$
\begin{aligned}
V(r, \theta) & =\frac{\sigma}{\epsilon_{0}}\left[R+r P_{1}(\cos \theta)+\frac{1}{2 R} r^{2} P_{2}(\cos \theta)+\ldots\right] \\
& =\frac{\sigma R}{2 \epsilon_{0}}\left[1+\left(\frac{r}{R}\right) \cos \theta+\frac{1}{4}\left(\frac{r}{R}\right)^{2}\left(3 \cos ^{2} \theta-1\right)+\ldots\right]
\end{aligned}
$$

