1. The induced dipole moment of a neutral atom is approximately proportional to the external electric field:
\[ p = \alpha E \]

The constant of proportionality \( \alpha \) is called atomic polarizability.

Now consider a hydrogen atom in its ground state. The electron cloud has a charge density
\[ \rho(r) = \frac{q}{\pi a^3} e^{-2r/a}, \]
where \( q \) is the charge of the electron and \( a \) is the Bohr radius (\( \equiv 0.5 \times 10^{-10} \) m).

Find the atomic polarizability of such an atom. (For comparison, the experimental value is \( 0.66 \times 10^{-30} \text{m}^3 \).)

[Hint: First calculate the electric field of the electron cloud, \( E_e(r) \). The nucleus will be shifted from \( r = 0 \) to its equilibrium position \( d \) where \( E_e = E_{\text{external}} \); expand the exponential in \( E_{\text{external}} \), assuming \( d \ll a \) and neglecting the higher order terms.]

2. The polarization in a linear dielectric is proportional to the field:
\[ P = \varepsilon_0 \chi_e E \quad (1) \]

If the material consists of atoms (or nonpolar molecules), the induced dipole moment of each atom is likewise proportional to the field:
\[ p = \alpha E \quad (2) \]

What is the relation between the atomic polarizability \( \alpha \) and the susceptibility \( \chi_e \)?

[Hint: Since \( P \) (dipole moment per unit volume) is \( p \) (dipole moment per atom) times \( N \) (number of atoms per unit volume), the obvious guess will be
\[ \varepsilon_0 \chi_e = N \alpha \quad (3) \]

However, there is a subtle problem, for the field \( E \) in Eq. 1 is the total macroscopic field in the medium, whereas the field in Eq. 2 is due to everything except the particular atom under consideration (by definition of atomic polarizability; see Problem 1); denote this field as \( E_{\text{else}} \). Assuming each atom to be a sphere of radius \( R \), show that
\[ E = \left( 1 - \frac{N \alpha}{3 \varepsilon_0} \right) E_{\text{else}} \quad (4) \]
Use this to obtain
\[\chi_e = \frac{N\alpha/\epsilon_0}{1 - N\alpha/3\epsilon_0},\]
or \[\alpha = \frac{3\epsilon_0}{N} \left( \frac{\epsilon_r - 1}{\epsilon_r + 2} \right)\] (5)

Solution

1. We calculate the field at radius \(r\) using Gauss’s law:
\[
\oint E \cdot da = \frac{Q_{\text{enc}}}{\epsilon_0}, \text{ or } E = \frac{Q_{\text{enc}}}{4\pi\epsilon_0 r^2}
\]
The total charge enclosed within a radius \(r\) is given by
\[
Q_{\text{enc}} = \int_0^r \rho d\tau = \frac{4\pi q}{\pi a^3} \int_0^r e^{-2r/a} r^2 d\bar{r}
\]
\[
= \frac{4q}{a^3} \left[ -\frac{a}{2} e^{-2r/a} \left( r^2 + a^2 \frac{r^2}{2} \right) \right]_0^r
\]
\[
= q \left[ 1 - e^{-2r/a} \left( 1 + 2r \frac{a^2}{a^2} \right) \right].
\]
(Note that \(Q_{\text{enc}}(r \to \infty) = q\.) So the field of the electron cloud is
\[
E_e(r) = \frac{q}{4\pi\epsilon_0 r^2} \left[ 1 - e^{-2r/a} \left( 1 + 2r \frac{a^2}{a^2} \right) \right].
\]
The proton in the nucleus will be shifted from \(r = 0\) to the point \(d\) where \(E_e = E\) (the external field):
\[
E = \frac{q}{4\pi\epsilon_0 d^2} \left[ 1 - e^{-2d/a} \left( 1 + 2 \frac{d^2}{a^2} \right) \right].
\]
Assuming \(d \ll a\), we expand the exponential in powers of \((d/a)\):
\[
e^{-2d/a} = 1 - \left( \frac{2d}{a} \right) + \frac{1}{2} \left( \frac{2d}{a} \right)^2 - \frac{1}{3} \left( \frac{2d}{a} \right)^3 + ...
\]
\[
= 1 - \frac{2d}{a} + 2 \left( \frac{d}{a} \right)^2 - \frac{4}{3} \left( \frac{d}{a} \right)^3 + ...
\]
\[
1 - e^{-2d/a} \left( 1 + \frac{2d}{a} + \frac{2d^2}{a^2} \right) = 1 - \left( 1 - \frac{2d}{a} + 2 \left( \frac{d}{a} \right)^2 - \frac{4}{3} \left( \frac{d}{a} \right)^3 + ... \right) \left( 1 + \frac{2d}{a} + \frac{2d^2}{a^2} \right)
\]
\[
= \frac{4}{3} \left( \frac{d}{a} \right)^3 + \text{ higher order terms.}
\]
So the external electric field is

\[ E = \frac{q}{4\pi\varepsilon_0 d^2} \left( \frac{4}{3} \frac{d^3}{a^3} \right) = \frac{p}{3\pi\varepsilon_0 a^3} \]

with \( p = qd \). So

\[ \alpha = 3\pi\varepsilon_0 a^3. \]

This result predicts \( \frac{\alpha}{4\pi\varepsilon_0} = \frac{3}{4}a^3 = \frac{3}{4}(0.5 \times 10^{-10})^3 = 0.09 \times 10^{-30} \text{m}^3 \), compared to the experimental value \( 0.66 \times 10^{-30} \text{m}^3 \). We can not expect a better result anyway in this classical picture. For a more sophisticated approach, see W.A. Bowers, *Am. J. Phys.* 54, 347 (1986).

2. The density of atoms is \( N = \frac{1}{(4/3)\pi R^3} \), assuming no voids in the material (idealized case). The macroscopic field \( E \) is \( E_{\text{self}} + E_{\text{else}} \), where \( E_{\text{self}} \) is the average field over the sphere due to the atom itself. From Eq. 3.105,

\[ E_{\text{self}} = -\frac{1}{4\pi\varepsilon_0 R^3} P \]

Now

\[ p = \alpha E_{\text{else}} \Rightarrow P = N\alpha E_{\text{else}}. \]

So

\[ E = -\frac{1}{4\pi\varepsilon_0 R^3} \alpha E_{\text{else}} + E_{\text{else}} = \left(1 - \frac{\alpha}{4\pi\varepsilon_0 R^3}\right) E_{\text{else}} \]

\[ = \left(1 - \frac{N\alpha}{3\varepsilon_0}\right) E_{\text{else}}. \]

So

\[ P = \frac{N\alpha}{1 - N\alpha/3\varepsilon_0} E = \varepsilon_0 \chi_e E, \]

and hence

\[ \chi_e = \frac{N\alpha}{1 - N\alpha/3\varepsilon_0}. \]

Solving for \( \alpha \):

\[ \alpha = \frac{3\varepsilon_0}{N} \frac{\chi_e}{3 + \chi_e}. \]

But \( \chi_e = \varepsilon_r - 1 \), so

\[ \alpha = \frac{3\varepsilon_0}{N} \left( \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right). \]

This equation is known as the Clausius-Mossotti formula, or, in its application to optics, the Lorentz-Lorenz equation.