PHYS 411

Discussion Class 7

19 October 2007

1. The *induced dipole moment* of a neutral atom is approximately proportional to the external electric field:

$$\mathbf{p} = \alpha \mathbf{E}$$

The constant of proportionality α is called **atomic polarizability**.

Now consider a hydrogen atom in its ground state. The electron cloud has a charge density

$$\rho(r) = \frac{q}{\pi a^3} e^{-2r/a},$$

where q is the charge of the electron and a is the Bohr radius ($\equiv 0.5 \times 10^{-10}$ m). Find the atomic polarizability of such an atom. (For comparison, the experimental value is $0.66 \times 10^{-30} \mathrm{m}^3$.)

[**Hint:** First calculate the electric field of the electron cloud, $E_e(r)$. The nucleus will be shifted from $r = 0$ to its equilibrium position d where $E_e = E_{\text{external}}$; expand the exponential in E_{external}, assuming $d \ll a$ and neglecting the higher order terms.]

2. The polarization in a linear dielectric is proportional to the field:

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} \tag{1}$$

If the material consists of atoms (or nonpolar molecules), the induced dipole moment of each atom is likewise proportional to the field:

$$\mathbf{p} = \alpha \mathbf{E} \tag{2}$$

What is the relation between the atomic polarizability α and the susceptibility χ_e?

[**Hint:** Since $\mathbf{P}$ (dipole moment per unit volume) is $\mathbf{p}$ (dipole moment per atom) times N (number of atoms per unit volume), the obvious guess will be

$$\epsilon_0 \chi_e = N\alpha \tag{3}$$

However, there is a subtle problem, for the field $\mathbf{E}$ in Eq. 1 is the *total macroscopic field* in the medium, whereas the field in Eq. 2 is due to everything *except* the particular atom under consideration (by definition of atomic polarizability; see Problem 1); denote this field as $\mathbf{E}_{\text{else}}$. Assuming each atom to be a sphere of radius R, show that

$$\mathbf{E} = \left(1 - \frac{N\alpha}{3\epsilon_0}\right) \mathbf{E}_{\text{else}} \tag{4}$$

Use this to obtain

$$\begin{aligned} \chi_e &= \frac{N\alpha/\epsilon_0}{1 - N\alpha/3\epsilon_0}, \\ \text{or} \quad \alpha &= \frac{3\epsilon_0}{N}\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right) \end{aligned} \tag{5}$$

Solution

1. We calculate the field at radius r using Gauss's law:

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}, \text{ or } E = \frac{Q_{\text{enc}}}{4\pi\epsilon_0 r^2}$$

The total charge enclosed within a radius r is given by

$$\begin{aligned} Q_{\text{enc}} &= \int_0^r \rho d\tau = \frac{4\pi q}{\pi a^3} \int_0^r e^{-2\tilde{r}/a} \tilde{r}^2 d\tilde{r} \\ &= \frac{4q}{a^3}\left[-\frac{a}{2} e^{-2\tilde{r}/a}\left(\tilde{r}^2 + a\tilde{r} + \frac{a^2}{2}\right)\right]_0^r \\ &= q\left[1 - e^{-2r/a}\left(1 + \frac{2r}{a} + \frac{2r^2}{a^2}\right)\right]. \end{aligned}$$

(Note that $Q_{\text{enc}}(r \to \infty) = q$.) So the field of the elctron cloud is

$$E_e(r) = \frac{q}{4\pi\epsilon_0 r^2}\left[1 - e^{-2r/a}\left(1 + \frac{2r}{a} + \frac{2r^2}{a^2}\right)\right].$$

The proton in the nucleus will be shifted from $r = 0$ to the point d where $E_e = E$ (the external field):

$$E = \frac{q}{4\pi\epsilon_0 d^2}\left[1 - e^{-2d/a}\left(1 + \frac{2d}{a} + \frac{2d^2}{a^2}\right)\right].$$

Assuming $d \ll a$, we expand the exponential in powers of (d/a):

$$\begin{aligned} e^{-2d/a} &= 1 - \left(\frac{2d}{a}\right) + \frac{1}{2}\left(\frac{2d}{a}\right)^2 - \frac{1}{3}\left(\frac{2d}{a}\right)^3 + ... \\ &= 1 - \frac{2d}{a} + 2\left(\frac{d}{a}\right)^2 - \frac{4}{3}\left(\frac{d}{a}\right)^3 + ... \\ 1 - e^{-2d/a}\left(1 + \frac{2d}{a} + \frac{2d^2}{a^2}\right) &= 1 - \left(1 - \frac{2d}{a} + 2\left(\frac{d}{a}\right)^2 - \frac{4}{3}\left(\frac{d}{a}\right)^3 + ...\right)\left(1 + \frac{2d}{a} + \frac{2d^2}{a^2}\right) \\ &= \frac{4}{3}\left(\frac{d}{a}\right)^3 + \text{ higher order terms.} \end{aligned}$$

So the external electric field is

$$E = \frac{q}{4\pi\epsilon_0 d^2}\left(\frac{4}{3}\frac{d^3}{a^3}\right) = \frac{p}{3\pi\epsilon_0 a^3}$$

with $p = qd$. So

$$\alpha = 3\pi\epsilon_0 a^3.$$

This result predicts $\frac{\alpha}{4\pi\epsilon_0} = \frac{3}{4}a^3 = \frac{3}{4}(0.5 \times 10^{-10})^3 = 0.09 \times 10^{-30}\mathrm{m}^3$, compared to the experimental value $0.66 \times 10^{-30}\mathrm{m}^3$. We can not expect a better result anyway in this *classical* picture. For a more sophisticated approach, see W.A. Bowers, *Am. J. Phys.* **54**, 347 (1986).

2. The density of atoms is $N = \frac{1}{(4/3)\pi R^3}$ assuming no voids in the material (idealized case). The macroscopic field $\mathbf{E}$ is $\mathbf{E}_{\text{self}} + \mathbf{E}_{\text{else}}$, where $\mathbf{E}_{\text{self}}$ is the average field ove the sphere due to the atom itself. From Eq. 3.105,

$$\begin{aligned}
& \mathbf{E}_{\text{self}} = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3} \\
\text{Now} \quad & \mathbf{p} = \alpha\mathbf{E}_{\text{else}} \Rightarrow \mathbf{P} = N\alpha\mathbf{E}_{\text{else}}. \\
\text{So} \quad & \mathbf{E} = -\frac{1}{4\pi\epsilon_0}\frac{\alpha}{R^3}\mathbf{E}_{\text{else}} + \mathbf{E}_{\text{else}} = \left(1 - \frac{\alpha}{4\pi\epsilon_0 R^3}\right)\mathbf{E}_{\text{else}} \\
& \quad = \left(1 - \frac{N\alpha}{3\epsilon_0}\right)\mathbf{E}_{\text{else}}. \\
\text{So} \quad & \mathbf{P} = \frac{N\alpha}{(1 - N\alpha/3\epsilon_0)}\mathbf{E} = \epsilon_0\chi_e\mathbf{E}, \\
\text{and hence} \quad & \chi_e = \frac{N\alpha}{(1 - N\alpha/3\epsilon_0)}. \\
\text{Solving for } \alpha: \quad & \alpha = \frac{3\epsilon_0}{N}\frac{\chi_e}{3 + \chi_e}.
\end{aligned}$$

But $\chi_e = \epsilon_r - 1$, so

$$\alpha = \frac{3\epsilon_0}{N}\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right).$$

This equation is known as the **Clausius-Mossotti** formula, or, in its application to optics, the **Lorentz-Lorenz** equation.