PHYS 411 Discussion Class 8

26 October 2007

1. Since parallel currents attract, it is reasonable to think that the current within a single wire should contract into a tiny concentrated stream along the axis. However in practice the current typically distributes itself quite uniformly over the wire. How do you account for this?

[Hint: Assume that the positive charges (having a uniform density ρ_+) are at rest, and the negative charges (having a uniform density ρ_-) move at speed v. Balance the magnetic attraction with the electric repulsion created by the accumulation of these mobile charges near the axis. You should be able to show that in equilibrium $\rho_- = -\rho_+\gamma^2$, where γ is the usual gamma factor, $(1 - v^2/c^2)^{-1/2}$ and $c = 1/\sqrt{\mu_0\epsilon_0}$ is the speed of light.]

2. A sphere of linear magnetic material is placed in an otherwise uniform magnetic field \mathbf{B}_0 . Find the new magnetic field inside the sphere.

[**Hint**: The formal (and rigorous) way to solve this problem is to set up the boundary conditions and to use the Laplace solution for the magnetic potential. Here is an easier alternative: \mathbf{B}_0 magnetizes the sphere and sets up a field \mathbf{B}_1 within it; \mathbf{B}_1 magnetizes the sphere by an additional amount and sets up an additional field \mathbf{B}_2 ; and so on. The total resultant field inside the sphere is the sum of these infinite number of fields. (The sum is of course convergent as it should yield the same answer!)]

Solution

1. The mobile charges do tend to line up near the axis, but the resulting concentration of (negative) charges sets up an electric field that repels away further accumulation. Equilibrium is reached when the electric repulsion on a mobile charge q balances the magnetic attraction:

$$\mathbf{F} = q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})] = 0 \Rightarrow \mathbf{E} = -(\mathbf{v} \times \mathbf{B})$$

Let us assume the current to be in the z direction: $\mathbf{J} = \rho_- v \hat{\mathbf{z}}$ (where ρ_- and v are both negative). The magnetic field is then given by

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \Rightarrow B 2\pi s = \mu_0 J \pi s^2$$
$$\Rightarrow \quad \mathbf{B} = \frac{\mu_0 \rho_- v s}{2} \hat{\phi}$$

The electric field is given by

$$\begin{split} \oint \mathbf{E} \cdot d\mathbf{a} &= \frac{1}{\epsilon_0} (\rho_+ + \rho_-) \pi s^2 l \\ \Rightarrow \quad \mathbf{E} &= \frac{1}{2\epsilon_0} (\rho_+ + \rho_-) s \hat{\mathbf{s}} \end{split}$$

In equilibrium,

$$\frac{1}{2\epsilon_0}(\rho_+ + \rho_-)s\hat{\mathbf{s}} = -\left[(v\hat{\mathbf{z}}) \times \left(\frac{\mu_0\rho_-vs}{2}\hat{\phi}\right)\right] = \frac{\mu_0}{2}\rho_-v^2s\hat{\mathbf{s}}$$
$$\Rightarrow \quad \rho_+ + \rho_- = \rho_-(\epsilon_0\mu_0v^2) = \rho_-\left(\frac{v^2}{c^2}\right)$$
$$\Rightarrow \quad \rho_+ = -\rho_-\left(1 - \frac{v^2}{c^2}\right) = -\rho_-/\gamma^2, \quad \text{or } \rho_- = -\gamma^2\rho_+.$$

In this naive model, the mobile negative charges fill a smaller inner cylinder, leaving a shell of positive (stationary) charge at the outside; see D.C. Gabuzda, Am. J. Phys. **61**, 360 (1993) for further discussion.

Note: For typical velocities the two charge densities are essentially unchanged by the current (since $\gamma \approx 1$). In plasmas, however, where the positive charges are also free to move, this so-called *pinch effect* can be very significant.

2. \mathbf{B}_0 magnetizes the sphere:

$$\mathbf{M}_0 = \chi_m \mathbf{H}_0 = \frac{\chi_m}{\mu_0(1+\chi_m)} \mathbf{B}_0.$$

This magnetization sets up a field within the sphere given by Eq. 6.16:

$$\mathbf{B}_1 = \frac{2}{3}\mu_0 \mathbf{M}_0 = \frac{2}{3}\kappa \mathbf{B}_0 \quad \text{(where } \kappa \equiv \frac{\chi_m}{1 + \chi_m}\text{)}.$$

Now \mathbf{B}_1 magnetizes the sphere an additional amount $\mathbf{M}_1 = \frac{\kappa}{\mu_0} \mathbf{B}_1$. This sets up an additional field in the sphere:

$$\mathbf{B}_2 = \frac{2}{3}\mu_0 \mathbf{M}_1 = \frac{2}{3}\kappa \mathbf{B}_1 = \left(\frac{2\kappa}{3}\right)^2 \mathbf{B}_0,$$

and so on. The total field is the sum of all these fields:

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1 + \mathbf{B}_2 + \dots$$

$$= \mathbf{B}_0 + \left(\frac{2\kappa}{3}\right) \mathbf{B}_0 + \left(\frac{2\kappa}{3}\right)^2 \mathbf{B}_0 + \dots$$

$$= \left[1 + (2\kappa/3) + (2\kappa/3)^2 + \dots\right] \mathbf{B}_0$$

$$= \frac{\mathbf{B}_0}{1 - 2\kappa/3} = \left(\frac{1 + \chi_m}{1 + \chi_m/3}\right) \mathbf{B}_0.$$