PHYS 411

Discussion Class 9

02 November 2007

1. A square loop, cut out of a thick sheet of aluminium, is placed so that the top portion is in a uniform magnetic field B (pointing into the page), and is allowed to fall under gravity. If the magnetic field is 1 T (a common laboratory field), find the terminal velocity of the loop. Find the velocity of the loop as a function of time. How long does it take to reach 90% of the terminal velocity? What would happen if you cut a tiny slit in the ring, breaking the circuit?

   [**Note:** The dimensions of the loop should cancel out; the only numbers you need are: resistivity of aluminium, $\rho = 2.8 \times 10^{-8} \Omega\text{m}$; mass density of aluminium, $\eta = 2.7 \times 10^3 \text{kg/m}^3$ and $g = 9.8 \text{m/s}^2$.]

2. The centripetal acceleration in the circular motion of an atomic electron (charge e) about the nucleus (charge Q) is provided by the Coulomb attraction. Now imagine that a tiny magnetic field dB is slowly turned on, perpendicular to the plane of the orbit. Show that the increase in kinetic energy imparted by the induced electric field is just right to sustain circular motion *at the same radius*.

   [**Note:** It is useful to know that the magnitude of the electric field induced due to a time-varying magnetic field in a circular region of radius r is $E = \frac{1}{2} r \frac{dB}{dt}$ (cf. Example 7.7).]

Solution

1. The induced e.m.f. in the loop is given by

$$\mathcal{E} = Blv = IR \Rightarrow I = \frac{Blv}{R}$$

This would produce an upward magnetic force (acting only on one side of the loop which is entirely inside the field (Figure 1):

$$F_m = IlB = \frac{B^2 l^2 v}{R}$$

This opposes the gravitational force acting downward: $F_g = mg$. Hence the net

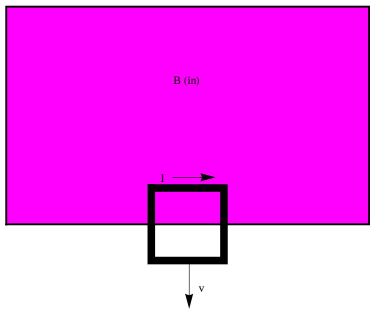

Figure 1: The colored region indicates the magnetic field **B** pointing into the page. The magnetic forces on the two vertical sides cancel each other, and the net magnetic force is on the horizontal side which is inside the region.

downward force is given by

$$\begin{aligned} m\frac{dv}{dt} &= mg - \frac{B^2 l^2}{R} v, \text{ or} \\ \frac{dv}{dt} &= g - \alpha v, \quad \text{where } \alpha \equiv \frac{B^2 l^2}{mR}. \end{aligned} \tag{1}$$

The terminal velocity is attained when the acceleration is zero:

$$g - \alpha v_t = 0 \Rightarrow v_t = \frac{g}{\alpha} = \frac{mgR}{B^2 l^2}.$$

The velocity as a function of time is obtained by solving the differential equation 1:

$$\frac{dv}{g - \alpha v} = dt \Rightarrow g - \alpha v = Ae^{-\alpha t}$$

At $t = 0,\ v = 0 \Rightarrow A = g$. Hence

$$\alpha v = g(1 - e^{-\alpha t}), \text{ or } v = \frac{g}{\alpha}(1 - e^{-\alpha t}) = v_t(1 - e^{-\alpha t}). \tag{2}$$

At 90% terminal velocity,

$$\frac{v}{v_t} = 0.9 = 1 - e^{-\alpha t} \Rightarrow t_{90\%} = \frac{1}{\alpha}\ln 10 = \frac{v_t}{g}\ln 10 \tag{3}$$

Now $m = \eta A 4l$, where A is the cross-sectional area and l is the length of each side. Also, $R = 4l\rho/A$. Hence

$$\begin{aligned} v_t &= \frac{4\eta A l g 4 l \rho}{A\sigma B^2 l^2} = \frac{16 g\eta\rho}{B^2} = \frac{16(9.8)(2.7\times 10^3)(2.8\times 10^{-8})}{1^2}\text{m/s} = 1.2 \text{ cm/s} \\ t_{90\%} &= \frac{1.2\times 10^{-2}}{9.8}\ln 10 = 2.8\times 10^{-3}\text{ s} = 2.8 \text{ ms} \end{aligned}$$

If the loop were cut, there would be no current flowing in the circuit, hence no magnetic force, and it would fall *freely* with acceleration g.

2. Initially,

$$\frac{mv^2}{r} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2} \Rightarrow E_{\text{kin}} = \frac{1}{2}mv^2 = \frac{1}{8\pi\epsilon_0}\frac{qQ}{r}$$

After the magnetic field is switched on, the electron circles in a new orbit, of radius r' and velocity v':

$$\frac{mv'^2}{r'} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r'^2} + qv'dB \Rightarrow E'_{\text{kin}} = \frac{1}{2}mv'^2 = \frac{1}{8\pi\epsilon_0}\frac{qQ}{r'} + \frac{1}{2}qv'r'dB$$

But $r' = r + dr$, so $\frac{1}{r'} = \frac{1}{r}\left(1 + \frac{dr}{r}\right)^{-1} \cong \frac{1}{r}\left(1 - \frac{dr}{r}\right)$, while $v' = v + dv$. So to first order, we have

$$E'_{\text{kin}} = \frac{1}{8\pi\epsilon_0}\frac{qQ}{r}\left(1 - \frac{dr}{r}\right) + \frac{1}{2}qvrdB$$

Hence the change in kinetic energy of the electron is

$$dE_{\text{kin}} = E'_{\text{kin}} - E_{\text{kin}} = \frac{qvr}{2}dB - \frac{1}{8\pi\epsilon_0}\frac{qQ}{r^2}dr \tag{4}$$

Now the induced electric field is

$$E = \frac{r}{2}\frac{dB}{dt}$$

Hence

$$m\frac{dv}{dt} = qE = \frac{qr}{2}\frac{dB}{dt}, \text{ or } mdv = \frac{qr}{2}dB.$$

The increase in kinetic energy is then given by

$$dE_{\text{kin}} = d\left(\frac{1}{2}mv^2\right) = mvdv = \frac{qvr}{2}dB \tag{5}$$

Comparing Eqs. 4 and 5, we conclude that $dr = 0$. Hence the electron will still circle at the same radius.