Blowout bifurcations and the onset of magnetic activity in turbulent dynamos

David Sweet,[1] Edward Ott,[1,*] John M. Finn,[2] Thomas M. Antonsen, Jr.,[1,*] and Daniel P. Lathrop[1]
[1]*Institute for Plasma Research and Department of Physics, University of Maryland, College Park, Maryland 20742*
[2]*Los Alamos National Laboratory, Los Alamos, New Mexico 87545*

(Received 23 June 2000; revised manuscript received 18 January 2001; published 21 May 2001)

The transition to magnetic-field self-generation in a turbulent, electrically conducting fluid is shown to exhibit intermittent bursting characterized by distinct scaling laws. This behavior is predicted on the basis of prior analysis of a type of bifurcation (called a *blowout bifurcation*) occurring in chaotic systems with an invariant phase space submanifold. The predicted scalings are shown to be consistent with numerical solutions of the governing magnetohydrodynamic equations, and implications for recently implemented experimental programs are discussed.

DOI: 10.1103/PhysRevE.63.066211 PACS number(s): 05.45.−a, 47.20.Ky, 47.65.+a

Active and ubiquitous large scale magnetic fields exist in stars and planets [1]. It is generally thought that these fields are generated by magnetohydrodynamic (MHD) dynamo action induced by the convection driven flow of an electrically conducting fluid (liquid metal or plasma) within these objects [1–3]. Several groups have initiated experimental programs to produce a laboratory-scale MHD dynamo [4,5]. Such experiments have the potential of ultimately allowing laboratory investigation of the basic dynamics and saturation mechanisms of naturally occurring planetary and stellar dynamos. In such experiments, as the forcing of the flow is increased, it is to be expected that there will be a transition from a situation where initial magnetic-field perturbations decay to zero to a situation where magnetic fields are self-generated by the flow. The fundamental question, which we address in this paper, is ''What is the likely character of this transition?'' In considering this question it is important to note that the magnetic Prandtl number (ratio of viscosity to magnetic diffusivity) is typically very small (e.g., of the order of 10^{-5} for liquid sodium). This has the consequence that, in unconstrained geometries [4,5], the fluid flow becomes turbulent well before the flow forcing is large enough for magnetic-field self-generation to occur. In this paper we present analytical evidence and numerical MHD computations showing that the essentially random dynamics of the turbulent fluid flow is likely to lead to extremely intermittent bursting of the magnetic activity just after the transition to dynamo action. Furthermore, we show that this bursting should obey definite scaling laws near transition, and we propose that these scalings may be measurable in future experiments. Beyond the specific interest for magnetic-field self-generation, our paper also provides an interesting physical context for the occurrence of a blowout bifurcation [6], which is a type of bifurcation recently studied in connection with chaotic dynamical systems in which chaos exists on an invariant submanifold in the state space of the system. Blowout bifurcations can be nonhysteretic (supercritical) or hysteretic (subcritical). In the nonhysteretic case they are characterized by a form of intermittent bursting that has been called on-off intermittency [7–11]. It is this type of intermittency that we find in our MHD computations.

Our numerical solutions of the nondimensionalized incompressible ($\nabla\cdot\mathbf{v}=0$) MHD equations,

$$\partial\mathbf{v}/\partial t+\mathbf{v}\cdot\nabla\mathbf{v}=-\nabla p+(\nabla\times\mathbf{B})\times\mathbf{B}+R^{-1}\nabla^2\mathbf{v}+\mathbf{F}, \quad (1)$$

$$\partial\mathbf{B}/\partial t+\mathbf{v}\cdot\nabla\mathbf{B}=\mathbf{B}\cdot\nabla\mathbf{v}+R_m^{-1}\nabla^2\mathbf{B}, \quad (2)$$

employ periodic boundary conditions in (x,y,z) with periodicity length 2π. The external stirring force is $\mathbf{F}(\mathbf{x})=[(\sin z+\cos y)\mathbf{x}_0+(\sin x+\cos z)\mathbf{y}_0+(\sin y+\cos x)\mathbf{z}_0]$. This ABC forcing has often been used to study aperiodic flows in three dimensions [12,13]. The system, Eqs. (1) and (2), is solved using a pseudospectral code in which $\mathbf{v}$ and $\mathbf{B}$ are expanded in Fourier modes, $\exp(i\mathbf{k}\cdot\mathbf{x})$, where $k_{x,y,z}$ are integers. These expansions are truncated at $|k_{x,y,z}|\leqslant K$. We investigate the solutions of (1) and (2) varying the magnetic Reynolds number R_m, keeping the fluid Reynolds number R fixed at $R=6.3$. We find that the transition to dynamo action occurs as R_m increases through a critical value $R_m=R_{mc}=7.9$. For $R_m\sim 7.9$ our tests indicate that $K=6$ gives converged results in the sense that increasing K past 6 does not lead to quantitative change in statistical behavior. We also find that even at $K=1$ [which does not represent a quantitatively faithful solution to Eqs. (1) and (2)] the correct qualitative behavior and scalings observed at $K=6$ still apply.

Figure 1(a) shows a plot of $\|\mathbf{B}\|\equiv[\int|\mathbf{B}(\mathbf{x},t)|^2d^3x]^{1/2}$ versus t from computations of Eqs. (1) and (2) with $(R_m-R_{mc})/R_{mc}=0.010$. We see that there are short intermittent bursts of magnetic activity separated by relatively long epochs where $\|\mathbf{B}\|$ is extremely small. Furthermore, viewing similar plots at R_m still closer to R_{mc}, we note that the rate of bursting is smaller closer to the transition. In addition, although the bursts are rarer for smaller $R_m-R_{mc}>0$, there is no noticeable difference in the burst amplitudes. The basic picture is that, as the transition is approached, $R_m-R_{mc}\rightarrow 0^+$, the average interburst time approaches infinity, but the typical burst amplitude remains virtually unchanged. As R_m-R_{mc} becomes larger, bursts become more and more frequent, eventually merging to yield a nonbursting, but chaotic, time variation of $\|\mathbf{B}\|$.

*Also with the Department of Electrical and Computer Engineering.

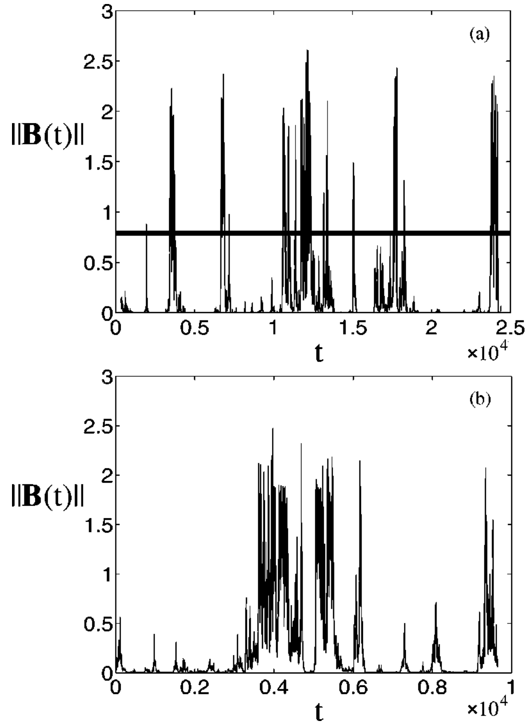

FIG. 1. (a) $\|\mathbf{B}\|$ versus t from our MHD computations at $(R_m - R_{mc})/R_{mc} = 0.010$. (b) $\|\mathbf{B}\|$ versus t for $(R_m - R_{mc})/R_{mc} = -0.015$ and an applied field of 1.6×10^{-4} in the x direction.

For $R_m < R_{mc}$, an initial small **B**-field perturbation eventually decays $\|\mathbf{B}\| \to 0$ as $t\to\infty$. Near $R_m - R_{mc} = 0^-$, however, the approach to $\|\mathbf{B}\| = 0$ can be very nonuniform in that several large bursts may sometimes appear before $\mathbf{B}=\mathbf{0}$ is approached. This has the consequence that, if a small constant external field is applied, sustained bursting occurs near transition even for $R_m < R_{mc}$. This is shown in Fig. 1(b) for $(R_m - R_{mc})/R_{mc} = -0.015$ and an applied field $\mathbf{B}_0 = B_0\mathbf{x}_0$, $B_0 = 1.6\times10^{-4}$. The issue of the effect of a small applied magnetic field is of particular importance for experiments due to the presence of the Earth's magnetic field and other stray fields in laboratory settings.

To understand the nature of the bursting process we note that near transition for most times the magnetic field is small. Therefore, we investigate the behavior of Eqs. (1) and (2) for **B** small. Note that $\mathbf{B}\equiv\mathbf{0}$ is consistent with Eqs. (1) and (2), and simply yields the usual magnetic-field-free Navier-Stokes equations with the given forcing $\mathbf{F}(\mathbf{x})$, boundary conditions, and fluid Reynolds number. We find that the magnetic-field-free solution is chaotic; i.e., **v** at any fixed point in space varies chaotically in time. The problem linearized about $\mathbf{B}=\mathbf{0}$ is called the kinematic dynamo problem [3] and consists of investigating the stability of our basic temporally chaotic magnetic-field-free flow to infinitesimal magnetic perturbations, $\delta\mathbf{B}(\mathbf{x},t)$. Note that the Lorentz force in Eq. (1), being quadratic in **B**, does not contribute in linear order, and thus $\delta\mathbf{v}=0$. Because **v** varies chaotically in time, the temporal behavior of $\|\delta\mathbf{B}\|$ does not approach an exponential. Rather, the instantaneous growth or damping of $\|\delta\mathbf{B}\|$ varies in an erratic manner in time. We can define a finite time growth rate of $\|\delta\mathbf{B}\|$ for the time interval T to $T+\tau$

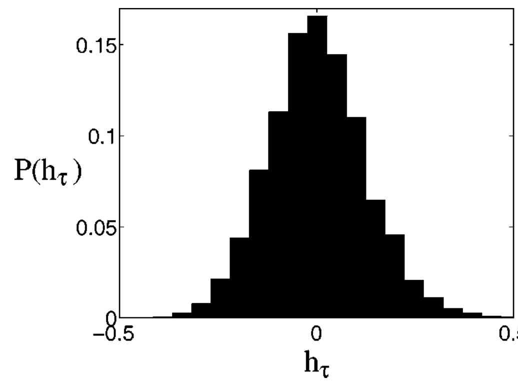

FIG. 2. $P(h_\tau)$ versus h_τ from our numerical computations with $\tau=32$ and $(R_m - R_{mc})/R_{mc} = 0.010$.

$$h_\tau = \tau^{-1}\log[\|\delta\mathbf{B}(\mathbf{x},T+\tau)\|/\|\delta\mathbf{B}(\mathbf{x},T)\|]. \tag{3}$$

For a randomly chosen value of T, with fixed τ and fixed $\delta\mathbf{B}(\mathbf{x},0)$, the finite time growth rate h_τ (also called a finite time Lyapunov exponent [14]) is a random variable. Figure 2 shows a histogram approximation to $P(h_\tau)$, the probability distribution function for h_τ, obtained from our kinematic dynamo computations with $\tau=32$. A qualitatively similar plot has been obtained in liquid metal dynamo experiments [5] (although, since the experiment is relatively far from the transition to self-field generation, all the measured h_τ values are negative). From the theory of blowout bifurcations the existence of spread in $P(h_\tau)$ determines the character of the bifurcation. Since this is present both in our simulation at $R\sim6.3$ and in the experiment with $R\sim10^7$, we expect similar qualitative transition behavior of the magnetic field in both cases [15].

Assuming ergodicity of the chaotic velocity field evolution, there is a well-defined infinite-time growth rate h, which is h_τ in the limit $\tau\to\infty$, where the limiting value h is the same for all values of T and almost all choices of the initial perturbation $\delta\mathbf{B}(\mathbf{x},0)$ [14]. For large τ, the width of the distribution $P(h_\tau)$ becomes narrower [14,16], approaching a delta function at h as $\tau\to\infty$,

$$\langle(h_\tau - \langle h_\tau\rangle)^2\rangle \sim 2D/\tau. \tag{4}$$

The quantity D in Eq. (4) plays an important role in the theoretical results to be discussed subsequently. We numerically estimate D as the slope of a plot of $(1/2)\tau^2\langle(h_\tau - \langle h_\tau\rangle)^2\rangle$ versus τ. For finite large τ we obtain an accurate numerical estimate of h by averaging h_τ over different runs with different initial conditions and T values. A plot of the estimated value of h versus R_m from our kinematic solutions of Eqs. (1) and (2) gives the critical value R_{mc} as that value of R_m at which h passes from $h<0$ (nondynamo) for $R_m < R_{mc}$ to $h>0$ (dynamo) for $R_m > R_{mc}$.

We now wish to quantitatively examine the scaling of statistical measures characterizing bursting time series such as those shown in Fig. 1. To do this we appeal to results previously derived for nonhysteretic blowout bifurcations [6,8–11] and the accompanying ''on-off intermittency.'' In a blowout bifurcation [6] one considers a dynamical system

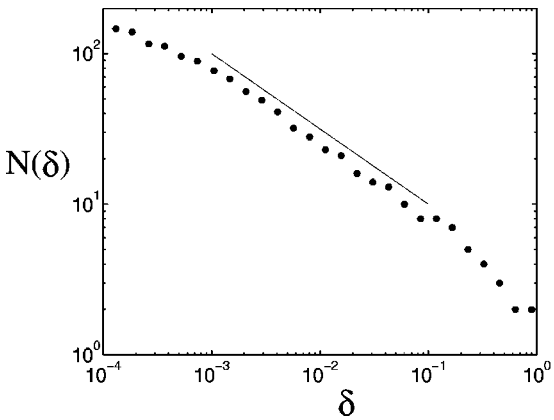

FIG. 3. Dots show $N(\delta)$ versus δ from our numerical computations of Eqs. (1) and (2) with $(R_m - R_{mc})/R_{mc} = 0.010$. The solid offset line has the theoretical slope of $-1/2$. The extent of this line $(10^{-3} < \delta < 10^{-1})$ indicates the expected scaling range $(Dt_*)^{-1} \ll \delta \ll D/(h^2 t_*)$.

with an invariant hypersurface of lower dimension than that of the full phase space of the system. Here by invariant we mean that if the state of the system is initially placed on the hypersurface, then the subsequent orbit remains on the hypersurface. Furthermore, it is assumed that initial conditions on the hypersurface are attracted to chaotic motion within the hypersurface. The blowout bifurcation refers to the loss of stability of the chaotic motion in the invariant hypersurface to perturbations transverse to the hypersurface. For our dynamo problem, the state of the system at time t is given by $\mathbf{v}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$. The situation $\mathbf{B} \equiv \mathbf{0}$ represents an invariant hypersurface in the full $\mathbf{v}-\mathbf{B}$ phase space, and, for the conditions of interest to us, motion on this hypersurface is chaotic; i.e., $\mathbf{v}(\mathbf{x},t)$, from the magnetic-field-free Navier-Stokes equations, varies irregularly in time. Thus the onset of magnetic dynamo action is a blowout bifurcation, and previous results for that bifurcation should be applicable. We test this below.

We consider the case where $1 \gg (R_m - R_{mc})/R_{mc} > 0$. To begin the discussion of the first scaling result, we introduce a threshold B_* by setting $B_* = \rho \|\mathbf{B}\|_{\max}$, where $\|\mathbf{B}\|_{\max}$ is the maximum value of $\|\mathbf{B}\|$ over the length of a long (i.e., many burst) computational time series, and $0 < \rho < 1$ is some order of one fraction. (The choice of the threshold is somewhat arbitrary, but the results are insensitive to this choice.) We define the burst times t_j as the instants when $\|\mathbf{B}\| = B_*$, $d\|\mathbf{B}\|/dt > 0$. Imagine that we obtain burst times for a long computational run, $0 \leq t \leq t_*$. We then normalize these times to the length of the run, $u_j = t_j/t_*$, so that u_j is in the interval (0,1). The claim is that, in the double limit $t_* \to \infty$ followed by $R_m - R_{mc} \to 0^+$, the set $\{u_j\}$ approaches a fractal of dimension $d = 1/2$. In particular, the number $N(\delta)$ of δ length intervals needed to cover the set $\{u_j\}$ scales as [10] $N(\delta) \sim \delta^{-1/2}$, for $(Dt_*)^{-1} \ll \delta \ll D/(h^2 t_*)$. Figure 3 shows that this prediction is well satisfied by our MHD computations. [Here B_* is given by the horizontal line in Fig. 1(a).]

As a test of the second scaling result, Fig. 4 shows a plot of the time average of $\|\mathbf{B}\|$ (denoted $\langle \|\mathbf{B}\| \rangle$) versus $(R_m - R_{mc})$. This plot yields results consistent with the expected linear dependence [9]

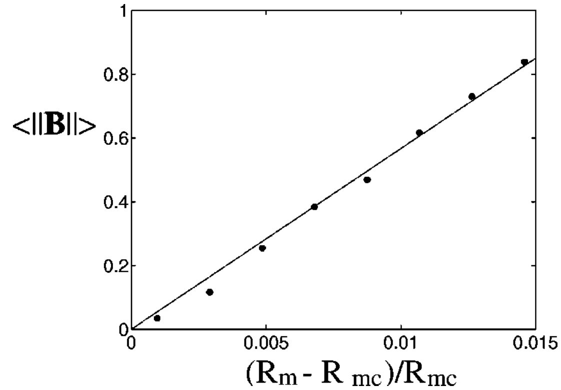

FIG. 4. $\langle \|\mathbf{B}\| \rangle$ versus $(R_m - R_{mc})/R_{mc}$ from our numerical computations of Eqs. (1) and (2) using truncation at $K=1$ ($K=6$ for Figs. 1–3). We use $K=1$ for this plot because it requires many long runs, and our computer resources are not sufficient for producing such a plot using $K=6$.

$$\langle \|\mathbf{B}\| \rangle \sim R_m - R_{mc}, \tag{5}$$

for small $(R_m - R_{mc})/R_{mc} > 0$. Since the character of $\|\mathbf{B}\|$ versus t is that of bursts up to $\|\mathbf{B}\|_{\max}$ with $\|\mathbf{B}\|_{\max}$ essentially constant for R_m close to R_{mc}, the scaling in Eq. (5) implies that the mean burst frequency becomes smaller and smaller, approaching zero as $(R_m - R_{mc}) \to 0^+$.

We have also tested various other predicted scalings for dynamo onset, and our numerical results are consistent with all of them [17]. They include power-law scaling ranges for the frequency (ω) power spectrum of $\|\mathbf{B}(t)\|$ ($\sim \omega^{-1/2}$) [8,10], the interburst time ($\mathbf{\Delta} = t_{j+1} - t_j$) probability distribution function ($\sim \mathbf{\Delta}^{-3/2}$) [11], and the probability distribution function of $\|\mathbf{B}\|$ for $\|\mathbf{B}\| \ll \|\mathbf{B}\|_{\max}$ ($\sim \|\mathbf{B}\|^{-\gamma}$, $\gamma = h/D - 1$).

Bursting occurs also in the case where $R_m < R_{mc}$, provided a small field is applied to the system; see Fig. 1(b). Scaling laws are predicted for this case [10,11], and we have shown that they hold for our numerical solutions for Eqs. (1) and (2) [17]. As an example, one can plot $N(\delta)$ versus δ for such a case. Again, as in the case $R_m > R_{mc}$, the fractal dimension $d = 1/2$ is expected with the difference that now the range of δ in which $N(\delta) \sim \delta^{-1/2}$ applies is limited by the size B_0 of the small applied field [10], $(Dt_*)^{-1} \ll \delta \ll (\ln nB_0^2)/(Dt_*)$.

In experiments, small external fields will be present. Thus there might be concern as to how to deduce a parameter transition value (e.g., our R_{mc}) from data. Scalings such as that shown in Fig. 4 and Eq. (5) can be useful in this regard. In particular, in the absence of a stray field $\langle \|\mathbf{B}\| \rangle = 0$ for $R_m < R_{mc}$ and $\langle \|\mathbf{B}\| \rangle \simeq k(R_m - R_{mc})$ for $R_m > R_{mc}$, where k is a constant, while, with stray fields, the transition of $\langle \|\mathbf{B}\| \rangle$ to zero as $R_m - R_{mc}$ decreases through zero is rounded. Nevertheless, as R_m increases, the behavior of $\langle \|\mathbf{B}\| \rangle$ with R_m should approach the predicted approximate linearity as $\langle \|\mathbf{B}\| \rangle$ becomes sufficiently large compared to the stray field. This linear portion of the curve can then be extrapolated to zero $\langle \|\mathbf{B}\| \rangle$ to estimate the critical value of the transition parameter.

Finally, we note that all of our discussion has been in the context of nonhysteretic (supercritical) blowout bifurcations. This is because our MHD computations yield this type of bifurcation. It is possible, however, that different forcing, boundary conditions, or geometry could yield a hysteretic (subcritical) blowout bifurcation. In such a case there are other scaling phenomena that are associated with a blowout bifurcation [16] and these would be expected to apply to a hysteretic dynamo transition.

This work was supported by the Office of Naval Research (Physics), and by the NSF (Contract Nos. EAR-9796295, EAR-9903958, EAR-9903162, and DMR-9896037). We thank Parvez Guzdar and Nicholas Peffley for useful discussions.

[1] H. F. Moffatt, *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge University, Cambridge, England, 1978); F. Krause and K. -H. Rädler, *Mean-Field Magnetohydrodynamics and Dynamo Theory* (Pergamon, Oxford, 1980); E. N. Parker, *Cosmical Magnetic Fields* (Clarendon, Oxford, 1979).

[2] S. Rüdiger, F. Feudel, and N. Seehafer, Phys. Rev. E **57**, 5533 (1998); G. A. Glatzmaier and P. H. Roberts, Nature (London) **377**, 203 (1995); G. A. Glatzmaier and P. H. Roberts, Phys. Earth Planet. Inter. **91**, 63 (1995); R. Kleva and J. F. Drake, Phys. Plasmas **2**, 4455 (1995).

[3] S. Childress and A. D. Gilbert, *Stretch, Twist, Fold: The Fast Dynamo* (Springer-Verlag, New York, 1995); B. J. Bayly and S. Childress, Geophys. Astrophys. Fluid Dyn. **44**, 211 (1988); E. Ott, Phys. Plasmas **5**, 1636 (1998); J. M. Finn and E. Ott, Phys. Fluids **31**, 2992 (1988); C. Reyl, E. Ott, and T. M. Antonsen, Phys. Plasmas **3**, 2564 (1996); D. Galloway and U. Frisch, Geophys. Astrophys. Fluid Dyn. **36**, 53 (1986); M. L. Dudley and R. W. James, Proc. R. Soc. London, Ser. A **A425**, 407 (1989).

[4] These experiments are currently being performed in various groups around the world, including groups in Karlsruhe, Germany [U. Müller and R. Stieglitz, Naturwissenschaften **87**, 381 (2000)]; Riga, Latvia [A. Gailitis, *et al.*, Magnetohydrodynamics **23**, 349 (1987); and A. Gailitis, *et al.*, Phys. Rev. Lett. **84**, 4365 (2000)]; Cadarache, France [P. Odier, J.-F. Pinton, and S. Fauve, Phys. Rev. E **58**, 7397 (1998)]; Madison, Wisconsin [C. Forest, private communication]; Socorro, New Mexico [S. Colgate, private communication]; and College Park, Maryland [5]. The devices in Riga and Karlsruhe self-generate in constrained geometries in which internal walls severely limit turbulent fluctuations. The efforts in the other laboratories have relatively unconstrained stirred flows with typical hydrodynamic Reynolds numbers of $R \sim 10^7$. The fact that the experiment in Karlesruhe does not observe bursting is due to their constrained geometry and the consequent lack of large scale turbulent velocity fluctuations.

[5] N. Peffley, A. Cawthorne, and D. P. Lathrop, Phys. Rev. E **61**, 5287 (2000).

[6] E. Ott and J. C. Sommerer, Phys. Lett. A **188**, 39 (1994).

[7] E. Spiegel, N.Y. Acad. Sci. **357**, 305 (1981).

[8] H. Fujisaka *et al.*, Prog. Theor. Phys. **76**, 1198 (1986); H. Fujisaka and T. Yamada, *ibid.* **74**, 919 (1985); **75**, 1087 (1986).

[9] L. Yu, E. Ott, and Q. Chen, Phys. Rev. Lett. **65**, 2935 (1990); Physica D **53**, 102 (1991).

[10] S. C. Vankataramani *et al.*, Phys. Lett. A **207**, 173 (1995); Physica D **96**, 66 (1996).

[11] N. Platt *et al.*, Phys. Rev. Lett. **70**, 3498 (1994); J. F. Heagy *et al.*, Phys. Rev. E **49**, 1140 (1994).

[12] V. I. Arnold, Comptes Rendus Acad. Sci. Paris **261**, 17 (1965), T. Dombre, U. Frisch, J. M. Greene, M. Hénon, A. Mehr, and A. Soward J. Fluid Mech. **167**, 353 (1986).

[13] O. Podvigina and A. Pouquet, Physica D **75**, 471 (1994); O. Podvigina, *ibid.* **128**, 250 (1999).

[14] E. Ott, *Chaos in Dynamical Systems* (Cambridge University, Cambridge, 1993), Sec. 9.4.

[15] The expectation of similar behavior in the simulations and the much higher R experiments is also supported by the following argument. At onset, $R_m = R_{mc}$, the tendency to self-generate magnetic field is balanced by magnetic diffusion [the term $R_m^{-1}\nabla^2\mathbf{B}$ in Eq. (2)]. Shorter spatial magnetic-field scales are more rapidly damped by diffusion. Consequently, we expect that, at onset, the magnetic field varies predominantly at large spatial scales L. Thus low-pass spatial filtering of $\mathbf{B}$ for wave numbers $k < k_0$, $k_0 L \gg 1$, leaves $\mathbf{B}$ approximately unchanged. Applying such a filter to Eq. (2) approximately reproduces Eq. (2) with $\mathbf{v}$ replaced by its filtered version $\tilde{\mathbf{v}}$. High fluid Reynolds number R is characterized by the creation, via turbulent cascade, of high wave-number components of $\mathbf{v}$ that are not present at lower R. The point is that high R dynamo action is essentially driven by the spatial low-pass filtered flow component $\tilde{\mathbf{v}}$ whose spatial variation is similar to that of flows with much lower R. Consequently, we expect our simulations to exhibit magnetic-field behavior similar to that of experiments at much higher R.

[16] E. Ott *et al.*, Physica D **76**, 384 (1994).

[17] D. Sweet *et al.*, Phys. Plasmas (to be published).